MATHEMATICS

CERTAIN DUAL SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS *)

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ABSTRACT

By applying a generalization of the multiplying factor technique employed by several earlier writers (cf. [3], [4] and [5]), an exact solution is obtained for the dual Laguerre series equations (1) and (2) below. Also computed are values of these series on the intervals over which they are not already specified.

1. INTRODUCTION AND PRELIMINARY RESULTS

In the present paper we give an exact solution of the dual series equations

$$(1)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+p+1)} L_{n+p}^{(\alpha)}(x) = f(x), \qquad 0 \leq x < y,$$

$$(2)\qquad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{(\sigma)}(x) = g(x), \qquad y < x < \infty,$$

where $\alpha+\beta+1>\beta>1-m$, $\sigma+1>\alpha+\beta>0$, m is a positive integer, p is an arbitrary non-negative integer,

$$(3)\qquad L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \qquad n=0, 1, 2, \ldots,$$

is the Laguerre polynomial of order α and degree n in x, and $f(x)$ and $g(x)$ are prescribed functions. The method used is a generalization of the multiplying factor technique which indeed was employed recently by Lowndes [3], Srivastava [5], and others (cf., e.g., [4]) to solve various special cases of the dual series equations (1) and (2). For a systematic account of the available techniques of solving these special cases (except possibly the case $\sigma=\alpha$ considered in [4]) the reader may be referred to Srivastava's paper [6].

We also compute the values of the series (1) and (2) on the intervals

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(y, ∞) and $[0, y)$, respectively, that is, on those intervals over which their values are not already specified.

The following results will be required in our investigation.

(i) The orthogonality property of the Laguerre polynomials given by [2, p. 292 (2)] and [op. cit., p. 293 (3)]:

$$\int_0^\infty e^{-x} x^\alpha L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{mn}, \quad \alpha > -1, \tag{4}$$

where δ_{mn} is the Kronecker delta.

(ii) Formula (27), p. 190 of reference [1] in the form:

$$\frac{d^m}{dx^m} \{x^{\alpha+m} L_n^{(\alpha+m)}(x)\} = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} x^\alpha L_n^{(\alpha)}(x). \tag{5}$$

(iii) The following forms of the known results [1, p. 191 (30)] and [2, p. 405 (20)]:

$$\int_0^\xi x^\alpha (\xi - x)^{\beta-1} L_n^{(\alpha)}(x)\, dx = \frac{\Gamma(\alpha+n+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+n+1)} \xi^{\alpha+\beta} L_n^{(\alpha+\beta)}(\xi), \tag{6}$$

where $\alpha > -1$, $\beta > 0$, and

$$\int_\xi^\infty e^{-x}(x-\xi)^{\beta-1} L_n^{(\alpha)}(x)\, dx = \Gamma(\beta) e^{-\xi} L_n^{(\alpha-\beta)}(\xi), \tag{7}$$

where $\alpha + 1 > \beta > 0$.

2. SOLUTION OF EQUATIONS (1) AND (2)

Multiplying equation (1) by $x^\alpha(\xi - x)^{\beta+m-2}$, where m is a positive integer, equation (2) by $e^{-x}(x-\xi)^{\sigma-\alpha-\beta}$, integrating them with respect to x over the intervals $(0, \xi)$ and (ξ, ∞), respectively, we find, on using (6) and (7), that

$$\left\{ \begin{aligned} &\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+m+n+p)} L_{n+p}^{(\alpha+\beta+n-1)}(\xi) \\ &\qquad = \frac{\xi^{-\alpha-\beta-m+1}}{\Gamma(\beta+m-1)} \int_0^\xi x^\alpha (\xi-x)^{\beta+m-2} f(x)\, dx, \end{aligned} \right. \tag{8}$$

where $0 < \xi < y$, $\alpha > -1$, $\beta + m > 1$, and

$$\left\{ \begin{aligned} &\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{(\alpha+\beta-1)}(\xi) \\ &\qquad = \frac{e^\xi}{\Gamma(\sigma-\alpha-\beta+1)} \int_\xi^\infty e^{-x}(x-\xi)^{\sigma-\alpha-\beta} g(x)\, dx, \end{aligned} \right. \tag{9}$$

where $y < \xi < \infty$, $\sigma + 1 > \alpha + \beta > 0$.

Now multiply equation (8) by $\xi^{\alpha+\beta+m-1}$, differentiate both sides m times with respect to ξ, and use the formula (5); we thus find that

$$(10)\quad \left\{\begin{aligned} &\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{(\alpha+\beta-1)}(\xi) \\ &\qquad = \frac{\xi^{-\alpha-\beta+1}}{\Gamma(\beta+m-1)} \frac{d^m}{d\xi^m} \int_0^{\xi} x^{\alpha}(\xi-x)^{\beta+m-2} f(x)\, dx, \end{aligned}\right.$$

where $0<\xi<y$, $\alpha>-1$, and $\beta+m>1$.

The first members of equations (9) and (10) are now identical, and an application of the orthogonality property (4) yields the desired solution of the dual series equations (1) and (2) in the form

$$(11)\quad \left\{\begin{aligned} A_n = &\frac{(n+p)!}{\Gamma(\beta+m-1)} \int_0^{y} e^{-\xi} L_{n+p}^{(\alpha+\beta-1)}(\xi) F(\xi)\, d\xi \\ &+ \frac{(n+p)!}{\Gamma(\sigma-\alpha-\beta+1)} \int_y^{\infty} \xi^{\alpha+\beta-1} L_{n+p}^{(\alpha+\beta-1)}(\xi)\, G(\xi)\, d\xi, \end{aligned}\right.$$

where $n,\ p \in \{0, 1, 2, \ldots\}$,

$$(12)\qquad F(\xi) = \frac{d^m}{d\xi^m} \int_0^{\xi} x^{\alpha} (\xi-x)^{\beta+m-2} f(x)\, dx,$$

and

$$(13)\qquad G(\xi) = \int_{\xi}^{\infty} e^{-x}(x-\xi)^{\sigma-\alpha-\beta} g(x)\, dx,$$

provided that $\alpha+\beta+1>\beta>1-m$ and $\sigma+1>\alpha+\beta>0$, m being a positive integer.

3. SPECIAL CASES

For $p=0$, the dual series equations (1) and (2) would reduce to those considered earlier by Srivastava [5], and indeed our solution (11) with $p=0$ is in complete agreement with Srivastava's solution [op. cit., p. 526, Eq. (10)].

On the other hand, if in our equation (11) we set $\sigma=\alpha$, and replace m by $m+1$, we shall obtain the solution to the dual equations discussed by Sharma and Shreshtha [4, p. 45, Eq. (3, 4)].

Finally, we observe that when $\sigma=\alpha$, $p=0$, $A_n=\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n)C_n$, $n\geqq 0$, the above equation (11) provides the solution to Lowndes' equations for $\alpha+\beta>0$, $1-m<\beta<1$, and indeed when $m=1$ the solutions are identical (cf. [3], p. 124, Eq. (12)).

4. VALUES OF SERIES (1) AND (2)

The values of the series (1) and (2) are not specified in the intervals

$y < x < \infty$ and $0 \leqq x < y$, respectively. In this section we show that these values can be determined without computing the coefficients A_n.

Let us suppose that

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{(\alpha)}(x) = \phi(x), \quad y < x < \infty \tag{14}$$

and

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{(\sigma)}(x) = \psi(x), \quad 0 \leqq x < y. \tag{15}$$

Then using (11) in (14) we get

$$\left\{ \begin{aligned} \phi(x) = {} & \frac{1}{\Gamma(\beta+m-1)} \int_0^y e^{-\xi} P(x, \xi) F(\xi) d\xi \\ & + \frac{1}{\Gamma(\sigma-\alpha-\beta+1)} \int_y^\infty \xi^{\alpha+\beta-1} P(x, \xi)\, G(\xi) d\xi, \end{aligned} \right. \tag{16}$$

where, for convenience,

$$P(x, \xi) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x)\, L_n^{(\alpha+\beta-1)}(\xi) - M(x, \xi) \tag{17}$$

and

$$M(x, \xi) = \sum_{n=0}^{p-1} \frac{n!}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x)\, L_n^{(\alpha+\beta-1)}(\xi), \tag{18}$$

it being understood that $M(x, \xi) = 0$ when $p = 0$.

By applying (4) and (7) in (17), it is easily verified that

$$P(x, \xi) = \frac{e^{\xi} x^{-\alpha} (x-\xi)^{-\beta}}{\Gamma(1-\beta)} H(x-\xi) - M(x, \xi), \tag{19}$$

where $M(x, \xi)$ is given by (18) and $H(t)$ denotes Heaviside's unit function. Substituting from (18) into (16) we obtain

$$\left\{ \begin{aligned} \phi(x) = {} & \frac{x^{-\alpha}}{\Gamma(1-\beta)} \left[\frac{1}{\Gamma(\beta+m-1)} \int_0^y (x-\xi)^{-\beta} F(\xi) d\xi \right. \\ & \left. + \frac{1}{\Gamma(\sigma-\alpha-\beta+1)} \int_y^\infty e^{\xi} \xi^{\alpha+\beta-1} (x-\xi)^{-\beta} G(\xi)\, d\xi \right] \\ & - \left[\frac{1}{\Gamma(\beta+m-1)} \int_0^y e^{-\xi} M(x, \xi)\, F(\xi)\, d\xi + \right. \\ & \left. + \frac{1}{\Gamma(\sigma-\alpha-\beta+1)} \int_y^\infty \xi^{\alpha+\beta-1} M(x, \xi)\, G(\xi)\, d\xi \right], \end{aligned} \right. \tag{20}$$

where $F(\xi)$, $G(\xi)$ and $M(x, \xi)$ are given by equations (12), (13) and (18), respectively.

Evidently, this last equation (20) yields the value of the series (1) when $y<x<\infty$.

In order to determine the value of $\psi(x)$ defined by (15), we consider the elementary result

$$e^{-x} L_n^{(\alpha)}(x) = -\frac{d}{dx}\{e^{-x} L_n^{(\alpha-1)}(x)\}, \tag{21}$$

which leads us by induction to the formula

$$e^{-x} L_n^{(\alpha)}(x) = (-1)^m \frac{d^m}{dx^m}\{e^{-x} L_n^{(\alpha-m)}(x)\}, \quad n \geqq 0, \tag{22}$$

for every positive integer m.

If we use (22) in (15), substitute for A_n from (11), and then interchange the order of integration and summation, we shall find that

$$\left\{\begin{aligned} e^{-x}\psi(x) = (-1)^m \frac{d^m}{dx^m}\Bigg\{ e^{-x}\Bigg[&\frac{1}{\Gamma(\beta+m-1)} \int_0^y e^{-\xi} Q(x, \xi) F(\xi)\, d\xi \\ &+ \frac{1}{\Gamma(\sigma-\alpha-\beta+1)} \int_y^\infty \xi^{\alpha+\beta-1} Q(x, \xi) G(\xi)\, d\xi\Bigg]\Bigg\}, \quad 0 \leqq x < y, \end{aligned}\right. \tag{23}$$

where

$$Q(x, \xi) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha+\beta+n)} L_n^{(\sigma-m)}(x)\, L_n^{(\alpha+\beta-1)}(\xi) - N(x, \xi) \tag{24}$$

and

$$N(x, \xi) = \sum_{n=0}^{p-1} \frac{n!}{\Gamma(\alpha+\beta+n)} L_n^{(\sigma-m)}(x)\, L_n^{(\alpha+\beta-1)}(\xi), \tag{25}$$

it being understood, as before, that this last sum is nil unless $p \geqq 1$.

Now using (4) and (6) to evaluate the first sum on the right-hand side of (24), we get

$$Q(x, \xi) = \frac{e^x \xi^{1-\alpha-\beta} (\xi-x)^{\alpha+\beta-\sigma+m-2}}{\Gamma(\alpha+\beta-\sigma+m-1)} H(\xi-x) - N(x, \xi), \tag{26}$$

where $N(x, \xi)$ is given by (25).

Thus we finally have

$$
(27)\quad \left\{\begin{aligned}
\psi(x)= &\frac{(-1)^m e^x}{\Gamma(\alpha+\beta-\sigma+m-1)}\,\frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(\beta+m-1)}\int_0^y e^{-\xi}\xi^{1-\alpha-\beta}\right.\\
&\left.\cdot(\xi-x)^{\alpha+\beta-\sigma+m-2}F(\xi)\,d\xi+\frac{1}{\Gamma(\sigma-\alpha-\beta+1)}\int_y^\infty(\xi-x)^{\alpha+\beta-\sigma+m-2}G(\xi)\,d\xi\right\}\\
&+(-1)^{m-1}e^x\frac{d^m}{dx^m}\left\{\frac{e^{-x}}{\Gamma(\beta+m-1)}\int_0^y e^{-\xi}N(x,\xi)\,F(\xi)\,d\xi\right.\\
&\left.+\frac{e^{-x}}{\Gamma(\sigma-\alpha-\beta+1)}\int_y^\infty \xi^{\alpha+\beta-1}N(x,\xi)\,G(\xi)\,d\xi\right\},\qquad 0\leqq x<y,
\end{aligned}\right.
$$

where $F(\xi)$, $G(\xi)$ and $N(x, \xi)$ are given by (12), (13) and (25), respectively.

REMARK 1. For $\sigma=\alpha$ and $p=0$, equations (20) and (27) with $m=1$ were given by Lowndes [3, p. 126, Eqs. (21) and (26)].

REMARK 2. In their special case $\sigma=\alpha$, if we replace m by $m+1$, equations (20) and (27) would reduce to the corresponding results in the Sharma-Shreshtha paper [4]. Notice, however, that the last result in reference [4] is in error which can easily be traced to their erroneous equation (4.12).

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